



## Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

SOLUTION OF A SYSTEM OF EQUATIONS OCCURRING IN  
DARBOUX'S *THÉORIE GÉNÉRALE DES SURFACES*.\*

By DR. T. CRAIG, Baltimore, Md.

The equations are, to notation *près*,

$$\begin{aligned} & x^2 + y^2 + z^2 = \text{const.} \\ \text{(A)} \quad & xx_1 + yy_1 + zz_1 = \text{const.} \end{aligned}$$

and we have also

$$\begin{aligned} & xx_2 + yy_2 + zz_2 = \text{const.}, \\ & x_1^2 + y_1^2 + z_1^2 = \text{const.} \\ & x_2^2 + y_2^2 + z_2^2 = \text{const.} \end{aligned}$$

It is required to find  $x, y, z$  from the first three of these. Darboux gives as the solution, without any indication of the process employed, the following :

$$\begin{aligned} x &= A_1x_1 + A_2x_2 + A_3(y_1z_2 - y_2z_1), \\ y &= A_1y_1 + A_2y_2 + A_3(z_1x_2 - z_2x_1), \\ z &= A_1z_1 + A_2z_2 + A_3(x_1y_2 - x_2y_1). \end{aligned}$$

A method for the solution of a general system of equations of the form (A) exists due to Bauer, but for the present case the following direct method, which is based on the most elementary geometrical considerations, seems to me preferable. The geometrical interpretation of the results and the notation is so obvious that it need not be referred to.

Write the equations in the form

$$\begin{aligned} x^2 + y^2 + z^2 &= C^2, \\ x_1^2 + y_1^2 + z_1^2 &= C_1^2, \end{aligned} \tag{1}$$

$$\begin{aligned} x_2^2 + y_2^2 + z_2^2 &= C_2^2, \\ xx_1 + yy_1 + zz_1 &= CC_1\lambda_1, \\ xx_2 + yy_2 + zz_2 &= CC_2\lambda_2; \end{aligned} \tag{2}$$

where  $C, C_1, C_2$  are arbitrary constants. Make now

$$\begin{aligned} x, y, z &= Ca, C\beta, C\gamma, \\ x_1, y_1, z_1 &= C_1a_1, C_1\beta_1, C_1\gamma_1, \\ x_2, y_2, z_2 &= C_2a_2, C_2\beta_2, C_2\gamma_2. \end{aligned} \tag{3}$$

---

\* t. I, p. 21; the equations preceding (6).

The three equations which we have ultimately to solve are now

$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= 1 \\ \alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1 &= \lambda_1 \end{aligned} \quad (4)$$

with

$$\begin{aligned} \alpha\alpha_2 + \beta\beta_2 + \gamma\gamma_2 &= \lambda_2 \\ \alpha_1^2 + \beta_1^2 + \gamma_1^2 &= 1 \\ \alpha_2^2 + \beta_2^2 + \gamma_2^2 &= 1. \end{aligned}$$

Define three new quantities  $x_3, y_3, z_3$  by the equations

$$\begin{aligned} x_3^2 + y_3^2 + z_3^2 &= C_3^2, \\ x_1x_3 + y_1y_3 + z_1z_3 &= 0, \\ x_2x_3 + y_2y_3 + z_2z_3 &= 0; \end{aligned} \quad (5)$$

also write

$$xx_3 + yy_3 + zz_3 = CC_3\lambda_3'$$

Making

$$x_3, y_3, z_3 = C_3\alpha_3, C_3\beta_3, C_3\gamma_3$$

we have

$$\begin{aligned} \alpha_3^2 + \beta_3^2 + \gamma_3^2 &= 1, \\ \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3 &= 0, \\ \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 &= 0, \\ \alpha\alpha_3 + \beta\beta_3 + \gamma\gamma_3 &= \lambda_3'. \end{aligned} \quad (6)$$

The second and third of these give

$$\frac{\alpha_3}{\beta_1\gamma_2 - \beta_2\gamma_1} = \frac{\beta_3}{\gamma_1\alpha_2 - \gamma_2\alpha_1} = \frac{\gamma_3}{\alpha_1\beta_2 - \alpha_2\beta_1}. \quad (7)$$

Write

$$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = \cos \theta; \quad (8)$$

square the terms in (7), add the numerators together and the denominators together, use the first of (6) and extract the square root of the result, this will be the common value of the ratios in (6), viz:

$$\frac{\alpha_3}{\beta_1\gamma_2 - \beta_2\gamma_1} = \frac{\beta_3}{\gamma_1\alpha_2 - \gamma_2\alpha_1} = \frac{\gamma_3}{\alpha_1\beta_2 - \alpha_2\beta_1} = \frac{1}{\sin \theta}, \quad (9)$$

or

$$\alpha_3 = \frac{\beta_1\gamma_2 - \beta_2\gamma_1}{\sin \theta}, \quad \beta_3 = \frac{\gamma_1\alpha_2 - \gamma_2\alpha_1}{\sin \theta}, \quad \gamma_3 = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\sin \theta}. \quad (10)$$

These satisfy identically the first three of equations (6). Take now the last two of equations (4) and the last of (6); these are

$$\begin{aligned} \alpha a_1 + \beta \beta_1 + \gamma \gamma_1 &= \lambda_1, \\ \alpha a_2 + \beta \beta_2 + \gamma \gamma_2 &= \lambda_2, \\ \alpha a_3 + \beta \beta_3 + \gamma \gamma_3 &= \lambda_3. \end{aligned} \quad (11)$$

Substituting the above values of  $a_3, \beta_3, \gamma_3$ , these become

$$\begin{aligned} \alpha (\beta_1 \gamma_2 - \beta_2 \gamma_1) + \beta (\gamma_1 a_2 - \gamma_2 a_1) + \gamma (a_1 \beta_2 - a_2 \beta_1) &= \lambda_3' \sin \theta = \lambda_3, \\ \alpha a_1 + \beta \beta_1 + \gamma \gamma_1 &= \lambda_1, \\ \alpha a_2 + \beta \beta_2 + \gamma \gamma_2 &= \lambda_2. \end{aligned} \quad (12)$$

Solving these we find

$$\begin{aligned} \alpha &= \frac{\lambda_3}{\sin^2 \theta} (\beta_1 \gamma_2 - \beta_2 \gamma_1) + \frac{\lambda_2}{\sin^2 \theta} (a_2 - a_1 \cos \theta) + \frac{\lambda_1}{\sin^2 \theta} (a_1 - a_2 \cos \theta), \\ \beta &= \frac{\lambda_3}{\sin^2 \theta} (\gamma_1 a_2 - \gamma_2 a_1) + \frac{\lambda_2}{\sin^2 \theta} (\beta_2 - \beta_1 \cos \theta) + \frac{\lambda_1}{\sin^2 \theta} (\beta_1 - \beta_2 \cos \theta), \\ \gamma &= \frac{\lambda_3}{\sin^2 \theta} (a_1 \beta_2 - a_2 \beta_1) + \frac{\lambda_2}{\sin^2 \theta} (\gamma_2 - \gamma_1 \cos \theta) + \frac{\lambda_1}{\sin^2 \theta} (\gamma_1 - \gamma_2 \cos \theta). \end{aligned} \quad (13)$$

Write

$$\begin{aligned} \frac{\lambda_1}{\sin^2 \theta} - \frac{\lambda_2 \cos \theta}{\sin^2 \theta} &= l_1, \quad \lambda_1 = l_1 + l_2 \cos \theta, \\ -\frac{\lambda_1 \cos \theta}{\sin^2 \theta} + \frac{\lambda_2}{\sin^2 \theta} &= l_2, \quad \lambda_2 = l_2 + l_1 \cos \theta. \end{aligned}$$

Equations (13) now become

$$\begin{aligned} \alpha &= l_1 a_1 + l_2 a_2 + \frac{\lambda_3}{\sin^2 \theta} (\beta_1 \gamma_2 - \beta_2 \gamma_1), \\ \beta &= l_1 \beta_1 + l_2 \beta_2 + \frac{\lambda_3}{\sin^2 \theta} (\gamma_1 a_2 - \gamma_2 a_1), \\ \gamma &= l_1 \gamma_1 + l_2 \gamma_2 + \frac{\lambda_3}{\sin^2 \theta} (a_1 \beta_2 - a_2 \beta_1). \end{aligned} \quad (14)$$

To determine  $\lambda_3$  (which with the auxiliaries  $a_3, \beta_3, \gamma_3$  is not arbitrary) we use the first of (4), viz.,

$$\alpha^2 + \beta^2 + \gamma^2 = 1. \quad (15)$$

This gives at once

$$\frac{\lambda_3}{\sin \theta} = \frac{\sqrt{1 - l_1^2 - l_2^2 - 2l_1l_2 \cos \theta}}{\sin \theta} =, \text{ say, } l_3. \quad (16)$$

So we have now

$$\begin{aligned} \alpha &= l_1\alpha_1 + l_2\alpha_2 + l_3(\beta_1\gamma_2 - \beta_2\gamma_1), \\ \beta &= l_1\beta_1 + l_2\beta_2 + l_3(\gamma_1\alpha_2 - \gamma_2\alpha_1), \\ \gamma &= l_1\gamma_1 + l_2\gamma_2 + l_3(\alpha_1\beta_2 - \alpha_2\beta_1). \end{aligned} \quad (17)$$

These are of the required form ; to get back to  $x, y, z$  write

$$l_1, l_2, l_3, = C_1m_1, C_2m_2, C_1C_2m_3;$$

then multiply each of (17) through by the arbitrary constant  $C$  and again write

$$Cm_1, Cm_2, Cm_3 = A_1, A_2, A_3,$$

where  $A_1, A_2, A_3$  are arbitrary constants, and we have finally

$$\begin{aligned} x &= A_1x_1 + A_2x_2 + A_3(y_1z_2 - y_2z_1), \\ y &= A_1y_1 + A_2y_2 + A_3(z_1x_2 - z_2x_1), \\ z &= A_1z_1 + A_2z_2 + A_3(x_1y_2 - x_2y_1), \end{aligned}$$

the required values. A number of modifications of the preceding process naturally suggest themselves, but it is not worth while to go into them.

BALTIMORE, *Sept.* 15, 1896.